SOLUTION OF PROBLEM 67.

BY G. W. HILL.

Assume the station as the origin of coordinates, the axis of x being directed toward the centre of the base, and that of z vertical. Let a be the radius of the base, c the altitude. The equation of the mountain's surface is then

$$a^2(c - z)^2 = c^2[(a - x)^2 + y^2].$$

The equation in terms of polar coordinates is obtained by putting

$$x = r\cos\theta\cos\omega, \qquad y = r\cos\theta\sin\omega, \qquad z = r\sin\theta,$$

and thus is $$r = 2ac\frac{c\cos\theta\cos\omega - a\sin\theta}{c^2\cos^2\theta - a^2\sin^2\theta}.$$

The element of volume of the mountain may be regarded as a rectangular solid whose sides are dr, $rd\theta$, $r\cos\theta\, d\omega$, and ρ being its density the element of mass is $\rho r^2\cos\theta\, dr\, d\theta\, d\omega$. Its attraction on the unit of mass at the station is $\rho\cos\theta\, dr\, d\theta\, d\omega$. From the symmetry of the cone it is plain that the component of the mountain's attraction in the direction of the axis of y is zero; and the vertical component which diminishes the intensity of gravity at the station may be neglected. The component in the direction of the axis of x is

$$X = \rho\int\int\int\cos^2\theta\cos\omega\, dr\, d\theta\, d\omega.$$

Integrating with respect to r, the limits are $r = 0$, and r=the value given by the equation of the surface. Thus

$$X = 2ac\rho\int\int\frac{c\cos\theta\cos\omega - a\sin\theta}{c^2\cos^2\theta - a^2\sin^2\theta}\cos^2\theta\cos\omega\, d\theta\, d\omega.$$

Next we integrate with respect to ω. As r must be always positive the limiting values of ω are the two roots of the equation $c\cos\omega = a\tan\theta$. Hence

$$X = 2ac\rho\int\left[\frac{c\cos^3\theta\cos^{-1}[(a\div c)\tan\theta]}{c^2\cos^2\theta - a^2\sin^2\theta} - \frac{\sin\theta\cos\theta}{\sqrt{(c^2\cos^2\theta - a^2\sin^2\theta)}}\right]d\theta.$$

The limits of integration are now from $\theta = 0$ to $\theta =$ the value given by the equation $a\tan\theta = c$. The second term within the brackets is integrable, and between the limits is $-\, a\div(a^2 + c^2)$. To simplify the first term revert to the variable ω, that is put $a\tan\theta = c\cos\omega$. Then

$$X = 2c\rho\left[\int_0^{\frac{\pi}{2}}\frac{\omega d\omega}{\sin\omega[1 + (c^2\div a^2)\cos^2\omega]^{\frac{3}{2}}} - \frac{a^2}{a^2 + c^2}\right].$$

The expression within the brackets is a function of $\frac{c}{a}$, calling it $F\left(\frac{c}{a}\right)$ we have

$$X = 2F\left(\frac{c}{a}\right)c\rho.$$

Now ρ' being the mean density and R the radius of the earth, the force of gravity g is

$$g = \frac{4\pi}{3}\rho' R,$$

and δ the deflection of the plumb-line is given by the equation

$$\tan\delta = \frac{X}{g} = \frac{3F(c \div a)}{2\pi}\,\frac{\rho}{\rho'}\,\frac{c}{R}.$$

The definite integral $\int_0^{\frac{\pi}{2}} \frac{\omega d\omega}{\sin\omega[1 + (c^2 \div a^2)\cos^2\omega]^{\frac{3}{2}}}$, it appears, must be computed by mechanical quadratures. Here is an opportunity to illustrate the use of the formula given at p. 9, Vol. II of the Analyst.

Divide the interval between the limits 0 and $\frac{1}{2}\pi$ into 9 equal parts; then $h = 10° = 0.1745241$. Compute the value of the function to be integrated multiplied by h for the middle of each of these parts, that is for $\omega = 5°, 15°, 25°, \ldots . 115°$. The three values beyond 90° are for the sake of the differences. We get

ω.	Δ^0.	ω.	Δ^0.
5°	0.1400956	65°	0.2094292
15	0.1432880	75	0.2327701
25	0.1497300	85	0.2594408
35	0.1595134	95	0.2899632
45	0.1727216	105	0.3258781
55	0.1893800	115	0.3705285

As the function integrated remains the same when the sign of ω is changed, all the odd orders of differences vanish for the argument $\omega = 0$. Then making $\Delta^{-1} = 0$ for the argument $\omega = 0$, by summing and differencing we get for the argument $\omega = 90°$

$$\Delta^{-1} = 1.6563687, \qquad \Delta^1 = +\,0.0305224,$$
$$\Delta^3 = +0.0015408, \qquad \Delta^5 = +\,0.0007833.$$

Thus the value of the definite integral by the formula is

$$= 1.6563687 + \frac{1}{24}\left(0.0305224\right) - \frac{17}{5760}\left(0.0015408\right) + \frac{367}{967680}\left(0.0007833\right)$$
$$= 1.6576363.$$

Consequently $F(0.4) = 0.7955673$, and the demanded deflection

$$\delta = 19''.21174.$$